

POWER-LAW SINGULARITIES IN STRING THEORY AND M-THEORY

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Abstract

We extend the definition of the Szekeres-Iyer power-law singularities to supergravity, string and M-theory backgrounds, and find that they are characterized by Kasner type exponents. The near singularity geometries of brane and some intersecting brane backgrounds are investigated and the exponents are computed. The Penrose limits of some of these power-law singularities have profiles $A \sim u^{-\gamma}$ for $\gamma \geq 2$. We find the range of the exponents for which $\gamma = 2$ and the frequency squares are bounded by $1/4$. We propose some qualitative tests for deciding whether a null or timelike spacetime singularity can be resolved within string theory and M-theory based on the near singularity geometry and its Penrose limits.

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1 INTRODUCTION

Szekeres and Iyer [1, 2] investigated the near singularity geometries of four-dimensional spherically symmetric solutions to the Einstein equations. They mostly focused on four-dimensional geometries with singularities of power-law type that arise during gravitational collapse, like the Lemaitre-Tolman-Bondi dust collapse metrics and Lifshitz-Khalatnikov singularities. It turns out that the geometry near the singularities can be described as

$$ds^2 = -2x^p dudv + x^q d\Omega_2^2 , \quad (1.1)$$

where

$$x = ku + \ell v , \quad \ell, k = 0, \pm 1 . \quad (1.2)$$

The Kasner type exponents p, q characterize the behaviour of the geometry near the singularity at $x = 0$.

It has been observed that string modes in plane waves with profiles $A \sim u^{-\gamma}$ for $\gamma < 2$ [3, 4] and for $\gamma = 2$ with frequency squares $\omega^2 < 1/4$ [5] can be extended across the singularity at $u = 0$; for similar results in field theory see [6, 7]. This is despite the fact plane waves with such profiles $\gamma > 0$ have a curvature singularity at $u = 0$ [8, 9]. An interpretation of this is that strings have a smooth propagation¹ in such backgrounds. Motivated by this, the authors of [10, 11] computed the Penrose limits of the near singularity geometries (1.1) and found that $A \sim u^{-\gamma}$, $\gamma \geq 2$, extending the results of [12, 13, 14, 15, 16]. In addition, they determined the range of the exponents for the wave profiles to behave as $A \sim u^{-2}$ with $\omega^2 < 1/4$. It was found that all geometries that satisfy (but not saturate) the dominant energy condition have Penrose limits with such profiles.

In this paper, we generalize the Szekeres-Iyer power-law singularities for string theory and M-theory backgrounds. This generalization applies to backgrounds with metrics and n -form field strengths which locally can be written as

$$\begin{aligned} ds^2 &= g_{mn}(y) dy^m dy^n + \sum_i G_i^2(y) ds_{(i)}^2 \\ F_n &= F_{mn}(y) dy^m \wedge dy^n \wedge \omega + F_m(y) dy^m \wedge \chi + F(y) \tau , \end{aligned} \quad (1.3)$$

where g is a two-dimensional Lorentz metric $m, n = 1, 2$, and $ds_{(i)}^2$ are smooth metrics and ω, χ, τ are forms on the Riemannian manifolds \mathcal{M}_i independent from the coordinates y^m . In addition, we assume that the spacetime has singularities of *codimension one*, ie every singularity is specified by a single function $C = C(y)$. Demanding that

¹However there are processes which are singular. For example for particular states there is an infinite string mode production near the singularity [8, 5].

the singularities are of power-law type, the near singularity geometry is characterized by Kasner type exponents both for the metric and the form-field strengths. The singularities that arise are timelike, spacelike or null.

A study of generic spacelike singularities in ten- and eleven-dimensional supergravities, following earlier work in general relativity [17], has revealed that their behaviour is chaotic [18]; for a review see [19]. This chaotic behaviour resembles that of a mixmaster universe where the Kasner exponents change infinite many times along independent spatial directions. Because of this our results for spacelike singularities are not generic. A similar analysis for weak null singularities in general relativity has been done in [20]. Nevertheless, the method we propose works well for some special backgrounds with spacelike and null singularities and also is applicable to backgrounds with timelike singularities.

We demonstrate that some brane backgrounds have singularities of power-law type and give their near singularity geometries. In particular, we compute the exponents of the near singularity geometries of the fundamental string and Dp-brane, $p \neq 3$, $p \leq 6$, backgrounds. It turns out that the singularities of fundamental string and Dp-branes, $p \leq 5$, are null while the singularity of the D6-brane is timelike. The metric of NS5-brane in the string frame, the D3-brane and M5-brane are not singular[21].

We also analyze the Penrose limits of the near singularity geometries of all power-law singularities. Some power-law singularities have diagonal plane wave profiles A which behave as $A \sim u^{-\gamma}$, $\gamma \geq 2$, where u is the affine parameter of a null geodesic and the singularity of the original spacetime is located at $u = 0$. We find the conditions on the exponents of these near singularity geometries for the wave profile to have behaviour $\gamma = 2$ and the frequency squares to be bounded by $\omega^2 \leq 1/4$. There are power-law singularities for which the Penrose limits have non-diagonal plane wave profiles. For these we give the plane wave metric in Rosen coordinates. Some of these Penrose limits may lead to homogeneous plane waves with rotation [22].

We propose a number of qualitative tests to decide whether a spacetime singularity can be resolved in string theory and M-theory. These tests mainly rely (i) on whether the near singularity geometry of a background can be identified with that of another singularity which has a well known description within string theory, (ii) on the assumption that the Penrose limits of a background can be taken in a regular way, and (iii) on whether string theory and M-theory is singular or well-defined at the Penrose limits. If a string background is singular but the singularity has a well-known interpretation, eg it has the near singularity geometry of a brane, and string theory is well-defined at all its Penrose limits, then these can serve as an indication that this singularity can be resolved within string theory. Though other tests should also be performed before it is

decided whether string theory is well-defined in such background, see eg [23, 24, 25]. Alternatively, if the near singularity geometry of a background does not have an interpretation within string theory and string theory is singular at a Penrose limit, eg string modes cannot propagate through the Penrose limit singularity, then we argue that such background is singular.

As an application of the above tests, we consider backgrounds with diagonal wave profiles $A \sim u^{-\gamma}$. We categorize such singularities into three types mild, marginal and severe. *Mild* singularities are those for which all the Penrose limits near the singularity have plane wave profiles $A \sim u^{-\gamma}$ with $0 < \gamma < 2$. *Marginal* singularities are those for which all the Penrose limits near the singularity have plane wave profiles $A \sim u^{-2}$. *Severe* singularities are those for which all the Penrose limits near the singularity have plane wave profiles $A \sim u^{-\gamma}$ with $\gamma > 2$. We shall argue that backgrounds which have timelike and null singularities of the marginal type with frequency squares $\omega^2 > 1/4$ and singularities of severe type may be singular in string theory. This is based on our result that the near singularity geometries of branes are either timelike or null and their Penrose limits have marginal singularities with frequency squares $\omega^2 \leq 1/4$ [12, 13, 14]. It is also known that string modes propagate across the mild [3] and marginal singularities [5] of plane waves with frequency squares $\omega^2 < 1/4$ but they are singular for the rest of planes waves with the above profiles [3]. It is also likely that these results generalize to spacelike singularities.

This paper is organized as follows: In section two, the definition of the near singularity geometries is given for (singular) string and M-theory backgrounds and describe how the near singularity geometry is characterized by exponents. In section three, we find the near singularity geometries of infinite planar brane solutions of ten- and eleven-dimensional supergravities. We also give the near singularity geometries of some intersecting brane configurations. In section four, we explain how the near singularity geometries of null and timelike power-law singularities and their Penrose limits can be used to provide some qualitative tests on whether string theory and M-theory is singular in certain backgrounds. In appendix A, we compute the Penrose limits of power-law singularities that arise in string and M-theory. We argue that some of them are associated with homogeneous singular plane waves with rotation. In appendix B, we describe the various Penrose limits that can be used for backgrounds with α' corrections.

2 CODIMENSION ONE POWER-LAW SINGULARITIES IN STRING THEORY AND M-THEORY

2.1 NEAR SINGULARITY METRICS

We shall extend the Szekeres-Iyer definition of near singularity geometries for power-law singularities to the class of string and M-theory backgrounds for which the metric can be written as

$$ds^2 = \gamma_{mn}(y)dy^m dy^n + \sum_i G_i^2(y)ds_{(i)}^2, \quad m, n = 0, 1. \quad (2.1)$$

In addition, we assume that some of the components of γ and G_i^2 vanish or are infinite at

$$C(y) = 0 \quad (2.2)$$

At such a hypersurface, the metric ds^2 can be singular. Since the singularity is specified by a single equation, it is of codimension one.

It is well-known that all two-dimensional metrics are conformally flat. Choosing the conformal gauge, we can write the metric ds^2 as

$$ds^2 = -2K^2(U, V)dUdV + \sum_i G_i^2(U, V)ds_{(i)}^2. \quad (2.3)$$

The transformations that preserve this form of the metric are two-dimensional conformal transformations in the coordinates (U, V) and the diffeomorphisms of \mathcal{M}_i .

The equation for the singularity can now be written as

$$C(U, V) = 0. \quad (2.4)$$

Suppose that in some conformal coordinates (U, V) the equation for the singularity can be written as

$$C(U, V) = m(U, V) \left(kf(U) + \ell h(V) \right)^\alpha = 0, \quad k, \ell = 0, \pm 1, \quad \alpha \in \mathbb{R}, \quad (2.5)$$

where m is a regular function and $m \neq 0, \infty$ for $kf(U) + \ell h(V) = 0$. If the singularity equation can be written as above², then there is always a coordinate x such that $x > 0$ and the singularity is located at $x = 0$. To show this, there are several cases to consider the following:

²This is not a strong assumption. For most singularities $C = 0$ can be solved using the inverse function theorem as $U = c(V)$ and then use a conformal transformation in V to find the coordinate x . However the assumption that the singularity equation is of the above form an advantage which enable us to describe singularities that are located at infinity.

(i) If $\eta = k\ell \neq 0$ and $\alpha > 0$, we perform the conformal transformation $u = f(U)$, $v = h(V)$ and define $x = kU + \ell V$. In this new coordinate, the singularity is located at $x = 0$. Moreover we choose k, ℓ such that $x > 0$.

(ii) If $\eta = k\ell \neq 0$ and $\alpha < 0$. We first define conformal coordinates $U' = f(U)$, $V' = h(V)$. It is now clear that the singularity lies at infinity, ie either at $U' = \pm\infty$ and/or at $V' = \pm\infty$. In the latter case, we perform the conformal transformation $u = U'$ and $v = 1/V'$. The equation for the singularity can be rewritten as

$$C(u, v) = m(u, \frac{1}{v}) \left(\frac{\ell v}{\eta uv + 1} \right)^{-\alpha}. \quad (2.6)$$

We define a new coordinate $x = \ell v$ and choose ℓ such that $x > 0$. The singularity is located at $x = 0$. The case that the singularity lies at $U' = \pm\infty$ can be treated in a similar way.

(iii) If $\eta = 0$, because say $k = 0$, and $\alpha > 0$, we again define conformal coordinates $v = h(V)$ and $u = U$ and $x = \ell v$. Again $x > 0$ with an appropriate choice of ℓ and the singularity is located at $x = 0$. The case with $k \neq 0$, $\ell = 0$ can be similarly treated.

(iv) if $\eta = 0$, say because $k = 0$, and $\alpha < 0$, we again define conformal coordinates $U' = U$, $V' = h(V)$. The singularity lies at infinity, $V' = \pm\infty$. In this case, we perform another conformal transformation $u = U'$, $v = 1/V'$ and define $x = \ell v$. Again $x > 0$ with an appropriate choice of ℓ and the singularity is located at $x = 0$. The case with $k \neq 0$, $\ell = 0$ can be similarly treated.

We have seen that for all types of singularities singularities we have considered, there are conformal coordinates (u, v) and a coordinate

$$x = ku + \ell v, \quad \eta = k\ell = 0, \pm 1, \quad x > 0 \quad (2.7)$$

which has the property that the singularity equation can be written as

$$C = \tilde{m}(u, v)x^\alpha, \quad \alpha > 0, \quad (2.8)$$

where \tilde{m} regular function and $\tilde{m} \neq 0, \infty$ at $x = 0$. Thus the singularity is located at $x = 0$. Singularities for which $x = ku + \ell v$ with $\eta = k\ell \neq 0$ are either spacelike ($\eta = 1$) or timelike ($\eta = -1$) while singularities for which $\eta = 0$ are null.

The metric in the u, v conformal coordinates is

$$ds^2 = -2L^2(u, v)dudv + \sum_i G_i^2(u, v)ds_{(i)}^2, \quad (2.9)$$

where the new conformal factor L^2 can be easily computed from K^2 and the conformal transformation necessary to bring the equation for the singularity in the form (2.8). The

components of the metric L^2, G_i^2 can be expressed as functions of the new coordinate x and either u , if $k = 0$ and $k, \ell \neq 0$, or v , if $\ell = 0$. Without loss of generality, we assume that L^2, G_i^2 can be expressed as functions of x, u . The analysis for the other case is similar. Since we are concerned with the behaviour of the metric near the singularity $x = 0$, we expand L^2, G_i^2 in power-series in x . We *demand* the singularity at $x = 0$ is of power-law type, ie L^2, G_i^2 have an expansion in x of the form

$$\begin{aligned} L^2 &= l(u)x^p + l_1(u)x^{p_1} + \dots \\ G_i^2 &= g_i(u)x^{w_i} + g_{i1}(u)x^{w_{i1}} + \dots \end{aligned} \quad (2.10)$$

for some regular functions $\{l, l_1, \dots\}$ and $\{g_i, g_{i1}, \dots\}$ and $l, g_i > 0$, where $p < p_1 < \dots$ and $w_i < w_{i1} < \dots$. Writing $L^2 = e^A$ and $G_i^2 = e^{B_i}$, we have that

$$\begin{aligned} A &= p \log x + \alpha(u, x) \\ B_i &= w_i \log x + \beta_i(u, x) \end{aligned} \quad (2.11)$$

where α, β_i are regular functions as $x \rightarrow 0$. The ‘near singularity geometry’ of the metric (2.9) is *defined* as

$$ds^2 = -2x^p du dv + \sum_i x^{w_i} ds_{(i)}^2. \quad (2.12)$$

The Kasner type exponents p, w_i characterize the power-law singularity. Observe that although we have started from a special class of metrics, the metric (2.12) is the most general power-law metric for spacelike, timelike and null singularities.

If one takes as the equation for the singularity $x = 0$, then there is no residual diffeomorphism invariance in the y^m coordinates that leaves both the form of the metric (2.12) and the equation of singularity invariant. However, there are residual transformations that only leave the form of the metric invariant. As a result, there is a relation between the Kasner type exponents p, w_i . We shall not pursue this further because the form of the metric (2.12) is sufficient for our purpose. Note that in the null case, the sign of the u, v component of the metric can change with a coordinate transformation so it has not an invariant meaning.

If $\eta \neq 0$, we define a new coordinate $y = ku - \ell v$ and rewrite the metric as

$$ds^2 = -\frac{1}{2}\eta x^p (dx^2 - dy^2) + \sum_i x^{w_i} ds_{(i)}^2, \quad \eta = k\ell. \quad (2.13)$$

It is now clear that the singularities with $\eta = 1$ are spacelike while those with $\eta = -1$ are timelike. As we have mentioned (2.13) does not describe the generic behaviour of spacelike singularities in supergravity theories because they are not of power-law type but exhibit mixmaster or chaotic behaviour [18, 19]. A similar analysis has been done

for weak null singularities in [20] and some applications have been found in [26, 27]. However (2.13) is general enough to describe all singularities of supergravity theories which are of power-law type.

In many examples that we shall investigate later, the metric (2.9) is of special form. This in turn leads to a near singularity metric which can be written as

$$d\bar{s}^2 = 2x^p dudv + \sum_i x^{w_i} ds^2(\mathbb{R}^{n_i}) + x^q d\Omega_m^2 . \quad (2.14)$$

The round m-sphere metric $d\Omega_m^2$ appears in many brane configurations as the sphere of the overall transverse space at infinity.

2.2 NEAR SINGULARITY FIELD STRENGTHS

M-theory and string theory backgrounds apart from the metric have other ‘active’ fields, like a dilaton or a form field strength. Therefore singular backgrounds are those for which either the metric or one of the other fields develops a singularity. To define the power-law singularities in the presence of other fields, we consider a background which has a non-vanishing n-form field strength F_n which is singular at the codimension one singularity (2.4). There are several definitions of what a singularity is in relation to the metric, for example one can take the definition that the spacetime is geodesically incomplete. In the case of other fields, one can also use various definitions. Here with singularity of a form field strength, we mean a region in spacetime that the form diverges. This is not an invariant definition but it will suffice for our purpose³. In the conformal coordinates (u, v) defined in the previous section, the n-form field strength (1.3) can be expanded as

$$F_n = F_1(u, v) du \wedge dv \wedge \omega + F_2(u, v) du \wedge \chi + F_3(u, v) dv \wedge \psi + F_4(u, v) \tau , \quad (2.15)$$

where ω, χ, ψ and τ are forms of appropriate degree in the remaining coordinates independent from u, v and $\{F_s, s = 1, \dots, 4\}$ are functions of u, v . (If F_n has degree one, then $\omega = 0$ and if F_n has degree zero $\omega = \chi = \psi = 0$ and τ is a scalar.) For power-law singularities, F_s have an expansion

$$F_s = f_s(u) x^{f_s} + f_{1s}(u) x^{f_{s1}} + \dots, \quad s = 1, \dots, 4 , \quad (2.16)$$

where $f_s < f_{s1} < \dots$, $f_s(u) \neq 0$. The form field strength of the near singularity geometry is defined as

$$\bar{F}_n = x^{f_1} du \wedge dv \wedge \omega + x^{f_2} du \wedge \chi + x^{f_3} dv \wedge \psi + x^{f_4} \tau \quad (2.17)$$

³An invariant definition of a singularity for F_n is to take the length of F_n to diverge.

in analogy with the definition of the near singularity metric.

The definition we have given for the near singularity geometry for power-law singularities captures the leading behaviour of the metric and form field strengths as one approaches the singularity. We have not verified that this leading order contribution solves the original field equations. Nevertheless, the above definition suffices for our purpose to investigate the qualitative properties of the singularities and their Penrose limits.

3 POWER-LAW SINGULARITIES AND BRANE CONFIGURATIONS

3.1 DP-BRANES

It is well-known that all Dp-brane backgrounds [30, 29, 30, 31], apart from the D3-brane, are singular in the string frame. The Dp-brane background can be written as

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}} ds^2(\mathbb{R}^{1,p}) + H^{\frac{1}{2}} (dr^2 + r^2 d\Omega_{8-p}^2) , & H &= 1 + \frac{Q_p}{r^{7-p}} , & p &\leq 6 \\ F_{p+2} &= d\text{vol}(\mathbb{R}^{1,p}) \wedge dH^{-1} \\ e^{2\phi} &= H^{\frac{3-p}{2}} , \end{aligned} \quad (3.1)$$

where F_{p+2} is the $(p+2)$ -form field strength, $p \neq 3$, and ϕ is the dilaton. The mass per unit volume of the Dp-brane, Q_p , does not contribute in the exponents of the singularity, so without loss of generality we can set $Q_p = 1$. The geometry near the singularity $r = 0$ is

$$ds^2 = r^{\frac{7-p}{2}} ds^2(\mathbb{R}^{1,p}) + r^{-\frac{7-p}{2}} (dr^2 + r^2 d\Omega_{8-p}^2) . \quad (3.2)$$

As we have explained in the previous section to find the metric near the singularity, we write the two-dimensional metric

$$ds_{(2)}^2 = -r^{\frac{7-p}{2}} dt^2 + r^{-\frac{7-p}{2}} dr^2 \quad (3.3)$$

in the conformal gauge. For this we first change coordinates as $r = w^\alpha$ which give

$$ds_{(2)}^2 = -w^{\frac{7-p}{2}\alpha} dt^2 + r^{-\frac{7-p}{2}\alpha+2\alpha-2} \alpha^2 dw^2 . \quad (3.4)$$

We then set $\alpha = -\frac{2}{5-p}$, $p \neq 5$, and the Dp-brane background can be written as

$$\begin{aligned} ds^2 &= w^{-\frac{7-p}{5-p}} \left(-dt^2 + \frac{4}{(5-p)^2} dw^2 + ds^2(\mathbb{R}^p) \right) + w^{\frac{3-p}{5-p}} d\Omega_{8-p}^2 \\ F_{p+2} &= -\frac{14-2p}{5-p} w^{-\frac{19-3p}{5-p}} d\text{vol}(\mathbb{R}^{1,p}) \wedge dw \\ e^{2\phi} &= w^{\frac{(3-p)(7-p)}{5-p}} . \end{aligned} \quad (3.5)$$

We further set $z = \frac{2}{|p-5|}w$ which gives

$$\begin{aligned}
ds^2 &= \left(\frac{|p-5|}{2}\right)^{-\frac{7-p}{5-p}} \left[z^{-\frac{7-p}{5-p}} \left(-dt^2 + dz^2 + ds^2(\mathbb{R}^p) \right) + \left(\frac{|p-5|}{2}\right)^2 z^{\frac{3-p}{5-p}} d\Omega_{8-p}^2 \right] \\
F_{p+2} &= -\left(\frac{14-2p}{5-p}\right) \left(\frac{|p-5|}{2}\right)^{-\frac{14-2p}{5-p}} z^{-\frac{19-3p}{5-p}} d\text{vol}(\mathbb{R}^{1,p}) \wedge dz \\
e^{2\phi} &= \left(\frac{|p-5|}{2}\right)^{\frac{(3-p)(7-p)}{5-p}} z^{\frac{(3-p)(7-p)}{5-p}} .
\end{aligned} \tag{3.6}$$

The above metric has singularities at $z = 0$ and $z = \infty$. For D6-branes, $r = \frac{1}{4}z^2$ and the singularity at $r = 0$ is located at $z = 0$. The singularity which is located at $z = \infty$ is an artifact of the approximation we made by replacing the harmonic function H with its near singularity value $1/r$ and so it is not a singularity of the D6-brane. Therefore, we have that the singularity is at $C = r = \frac{1}{4}z^2 = 0$, which can be solved by setting $z = 0$. Because of this, we set $x = z = u + v$, $t = v - u$, ($u = \frac{1}{2}(-t + x)$, $v = \frac{1}{2}(t + x)$) to find that the near singularity background is

$$\begin{aligned}
d\bar{s}^2 &= 2xdudv + xds^2(\mathbb{R}^6) + x^3d\Omega_2^2 \\
\bar{F}_{p+2} &= -x du \wedge dv \wedge d\text{vol}(\mathbb{R}^6) \\
e^{2\bar{\phi}} &= \frac{1}{8}x^3 .
\end{aligned} \tag{3.7}$$

The exponents of the power-law singularities of the D6-brane are

$$\begin{aligned}
ds^2 &: \text{ p } = 1 , \quad \text{ w } = 1 , \quad \text{ q } = 3 \\
F_8 &: \text{ f } = 1 \\
\phi &: \text{ d } = 3
\end{aligned} \tag{3.8}$$

The D6-brane singularity is a timelike singularity.

For the Dp-branes, $p < 5$, we have that $r = w^{-\frac{2}{5-p}} = \frac{|p-5|}{2} z^{-\frac{2}{5-p}}$ and so the singularity at $r = 0$ is located at $z = \infty$. To proceed define the conformal coordinates $U = \frac{1}{2}(-t + z)$, $V = \frac{1}{2}(t + z)$. In terms of these coordinates, the singularity equation C becomes

$$C = \frac{1}{z^{\frac{2}{5-p}}} = \frac{1}{(U+V)^{\frac{2}{5-p}}} = 0 . \tag{3.9}$$

The solutions to this equation are either $U = +\infty$ or $V = +\infty$. The two cases are symmetric and so without loss of generality we take the singularity to lie at $V = +\infty$. The other case whether the singularity lies at $U = +\infty$ leads to a near singularity geometry with the same exponents. As we have explained to bring the singularity from infinity to the origin, we perform the conformal transformation $U = u$, $V = 1/v$. After this transformation, the singularity equation is

$$C = \left(\frac{v}{uv+1}\right)^{\frac{2}{5-p}} = 0 . \tag{3.10}$$

and so we choose, $x = v$. The Dp-brane background becomes

$$\begin{aligned}
ds^2 &= 2\left(\frac{|p-5|}{2}\right)^{-\frac{7-p}{5-p}} \left[-2z^{-\frac{7-p}{5-p}} v^{-2} du dv + \frac{1}{2} z^{-\frac{7-p}{5-p}} ds^2(\mathbb{R}^p) + \frac{1}{2} \left(\frac{p-5}{2}\right)^2 z^{\frac{3-p}{5-p}} d\Omega_{8-p}^2 \right] \\
F_{p+2} &= (-1)^{p+1} 2 \left(\frac{14-2p}{5-p}\right) \left(\frac{|p-5|}{2}\right)^{-\frac{14-2p}{5-p}} z^{-\frac{19-3p}{5-p}} v^{-2} du \wedge dv \wedge d\text{vol}(\mathbb{R}^p) \\
e^{2\phi} &= \left(\frac{|p-5|}{2}\right)^{\frac{(3-p)(7-p)}{5-p}} z^{\frac{(3-p)(7-p)}{5-p}}, \tag{3.11}
\end{aligned}$$

where $z = u + \frac{1}{v}$. From this it is straightforward to see that the near singularity background, according to the definition given in the previous section, is

$$\begin{aligned}
d\bar{s}^2 &= 2x^{-\frac{3-p}{5-p}} du dv + x^{\frac{7-p}{5-p}} ds^2(\mathbb{R}^p) + x^{-\frac{3-p}{5-p}} d\Omega_{8-p}^2, \quad x = v \\
\bar{F}_{p+2} &= x^{\frac{9-p}{5-p}} du \wedge dv \wedge d\text{vol}(\mathbb{R}^p) \\
e^{2\bar{\phi}} &= x^{-\frac{(3-p)(7-p)}{5-p}}. \tag{3.12}
\end{aligned}$$

Note that in the case of null singularities as above the sign of the u, v component of the metric does not have an invariant meaning because it can change with a coordinate transformation $u \rightarrow -u$. The exponents of the power-law singularities of the Dp-branes, $p \neq 3, p \leq 4$, are

$$\begin{aligned}
ds^2 &: \quad p = -\frac{3-p}{5-p}, \quad w = \frac{7-p}{5-p}, \quad q = -\frac{3-p}{5-p} \\
F_{p+2} &: \quad f = \frac{9-p}{5-p}, \quad p \leq 4 \\
\phi &: \quad d = -\frac{(3-p)(7-p)}{5-p}. \tag{3.13}
\end{aligned}$$

The D3-brane solution, although it is included in the above analysis, it is non-singular at $x = 0$. Similarly, the metric of the NS5-brane [32] is not singular at $r = 0$, though the dilaton diverges at $r = 0$, and so we shall not investigate them further. So from the Dp-branes, it remains to investigate the D5-branes. In this case the transformation to conformal coordinates is $r = e^w$. Writing the background in conformal coordinates (U, V) , $w = U + V, t = V - U$, we have

$$\begin{aligned}
ds^2 &= 4e^{U+V} dU dV + e^{-U-V} ds^2(\mathbb{R}^5) + e^{U+V} d\Omega_3^2 \\
F_7 &= -4e^{2U+2V} dU \wedge dV \wedge d\text{vol}(\mathbb{R}^5) \\
e^{2\phi} &= e^{2U+2V}. \tag{3.14}
\end{aligned}$$

The equation for the singularity is

$$C = r = e^{U+V} = 0 \tag{3.15}$$

which has solutions either at $U = -\infty$ or at $V = -\infty$. We consider the case where the singularity is located at $V = -\infty$ and define new conformal coordinates $v = e^V$

and $u = e^U$. (The other case whether the singularity lies at $U = -\infty$ leads to a near singularity geometry with the same exponents.) In the (u, v) conformal coordinates the background can be rewritten as

$$\begin{aligned} ds^2 &= 4dudv + uvds^2(\mathbb{R}^5) + uv d\Omega_3^2 \\ F_7 &= -4uvdu \wedge dv \wedge d\text{vol}(\mathbb{R}^5) \\ e^{2\phi} &= u^2v^2 \end{aligned} \tag{3.16}$$

and the singularity is at $uv = 0$. Setting $x = v$, the near singularity geometry for the D5-brane is

$$\begin{aligned} d\bar{s}^2 &= 2dudv + xds^2(\mathbb{R}^5) + xd\Omega_3^2 \\ \bar{F}_7 &= xdu \wedge dv \wedge d\text{vol}(\mathbb{R}^5) \\ e^{2\bar{\phi}} &= x^2 \end{aligned} \tag{3.17}$$

The exponents of the power-law singularities of the D5-brane are

$$\begin{aligned} ds^2 &: p = 0, \quad w = 1, \quad q = 1 \\ F &: f = 1 \\ \phi &: d = 2. \end{aligned} \tag{3.18}$$

This concludes the computation of the near singularity geometries of the Dp-branes.

3.2 FUNDAMENTAL STRING

The fundamental string background [33] is

$$\begin{aligned} ds^2 &= H^{-1}(-dt^2 + d\rho^2) + dr^2 + r^2 d\Omega_7^2, \quad H = 1 + \frac{Q_F}{r^6} \\ F_3 &= d\text{vol}(\mathbb{R}^{1,1}) \wedge dH^{-1} \\ e^{2\phi} &= H^{-1}. \end{aligned} \tag{3.19}$$

Setting the mass per unit length, Q_F , of the string to $Q_F = 1$, the geometry near $r = 0$ is

$$\begin{aligned} ds^2 &= r^6(-dt^2 + d\rho^2) + dr^2 + r^2 d\Omega_7^2 \\ F_3 &= d\text{vol}(\mathbb{R}^{1,1}) \wedge dr^6 \\ e^{2\phi} &= r^6. \end{aligned} \tag{3.20}$$

Transforming the two-dimensional metric $-r^6 dt^2 + dr^2$ into the conformal gauge with the transformation $r = w^{-\frac{1}{2}}$, we find

$$ds^2 = w^{-3}(-dt^2 + \frac{1}{4}dw^2) + w^{-3}d\rho^2 + w^{-1}d\Omega_7^2$$

$$\begin{aligned}
F &= -3w^{-4}d\text{vol}(\mathbb{R}^{1,1}) \wedge dw \\
e^{2\phi} &= w^{-3} .
\end{aligned} \tag{3.21}$$

Performing the coordinate transformations $z = \frac{1}{2}w$, $U = \frac{1}{2}(-t + z)$ and $V = \frac{1}{2}(t + z)$, we rewrite the above metric as

$$\begin{aligned}
ds^2 &= \frac{1}{4}[2z^{-3}dUdV + \frac{1}{2}z^{-3}d\rho^2 + 2z^{-1}d\Omega_7^2] \\
F &= \frac{3}{4}z^{-4}dU \wedge dV \wedge d\rho \\
e^{2\phi} &= \frac{1}{8}z^{-3} .
\end{aligned} \tag{3.22}$$

To locate the singularity at $r = 0$ in the new coordinates, we note that $r = (2z)^{-\frac{1}{2}}$ and so it lies at $z = +\infty$. In the conformal coordinates U, V , the singularity at $z = +\infty$ is located at either $U = +\infty$ or $V = +\infty$. The two cases are symmetric and so without loss of generality we can take the singularity to lie at $V = +\infty$. The other case whether the singularity lies at $U = +\infty$ leads to a near singularity geometry with the same exponents. As we have explained, we perform the conformal transformation $U = u$ and $V = 1/v$ and set $x = v$. In (u, v) coordinates, the equation (3.22) can be written as

$$\begin{aligned}
ds^2 &= \frac{1}{4}[-2z^{-3}v^{-2}dudv + \frac{1}{2}z^{-3}d\rho^2 + 2z^{-1}d\Omega_7^2] \\
F_3 &= -\frac{3}{4}z^{-4}v^{-2}du \wedge dv \wedge d\rho \\
e^{2\phi} &= \frac{1}{8}z^{-3} ,
\end{aligned} \tag{3.23}$$

where $z = u + \frac{1}{v}$. From this it is easy to see that the near singularity geometry is

$$\begin{aligned}
d\bar{s}^2 &= 2xdudv + x^3d\rho^2 + xd\Omega_7^2 \\
\bar{F}_3 &= x^2du \wedge dv \wedge d\rho \\
e^{2\bar{\phi}} &= x^3 .
\end{aligned} \tag{3.24}$$

Therefore, the exponents of the fundamental string solution are

$$\begin{aligned}
ds^2 &: \quad p = 1 , \quad w = 3 , \quad q = 1 \\
F_3 &: \quad f = 2 \\
\phi &: \quad d = 3 .
\end{aligned} \tag{3.25}$$

In the M-theory both the membrane solution [34] and the five-brane [35] solution are non-singular at the position of the branes, $r = 0$. Because of this we shall not present the exponents for these cases.

3.3 INTERSECTING BRANES

There are several intersecting brane backgrounds [36, 37, 38]. Here we shall focus only in a few examples and present the near singularity geometries. In what follows we shall compute the near singularity geometries of the metrics. A class of intersecting brane configurations in string theory are those of a fundamental string orthogonally ending on a Dp-brane, $p < 6$. The metric of a supergravity solution which represents such a (delocalized) intersection in the string frame is

$$ds^2 = -H_D^{-\frac{1}{2}} H_F^{-1} dt^2 + H_D^{\frac{1}{2}} H_F^{-1} d\rho^2 + H_D^{-\frac{1}{2}} ds^2(\mathbb{R}^p) + H_D^{\frac{1}{2}} (dr^2 + r^2 d\Omega_{7-p}^2) \\ H_D = 1 + \frac{Q_D}{r^{6-p}}, \quad H_F = 1 + \frac{Q_F}{r^{6-p}}, \quad (3.26)$$

where H_D, H_F are the harmonic functions of the Dp-brane and the fundamental string, respectively. Setting the mass per volume parameters of the branes $Q_D = Q_F = 1$, the metric near the singularity at $r = 0$ is

$$ds^2 = -r^{\frac{3}{2}(6-p)} dt^2 + r^{\frac{6-p}{2}} (d\rho^2 + ds^2(\mathbb{R}^p)) + r^{-\frac{6-p}{2}} dr^2 + r^{\frac{p-2}{2}} d\Omega_{7-p}^2. \quad (3.27)$$

As in the case of branes, we perform a coordinate transformation $r = w^{-\frac{2}{10-2p}}$ to set the two-dimensional metric spanned by t, r in the conformal gauge. The above metric can then be rewritten as

$$ds^2 = w^{-\frac{3(6-p)}{10-2p}} \left(-dt^2 + \frac{4}{(10-2p)^2} dw^2 \right) + w^{-\frac{6-p}{10-2p}} (d\rho^2 + ds^2(\mathbb{R}^p)) + w^{-\frac{p-2}{10-2p}} d\Omega_{7-p}^2. \quad (3.28)$$

As in the case of Dp-branes, we define a new coordinate $z = \frac{2}{10-2p}w$, $p < 6$, and conformal coordinates $U = \frac{1}{2}(t + z), V = \frac{1}{2}(-t + z)$. It is clear that the singularity at $r = 0$ for $p \leq 4$ is located at either $U = +\infty$ or $V = +\infty$. The two cases are symmetric so we take that the singularity lies at $V = +\infty$. As we have explained to locate the singularity at the origin, we perform a conformal transformation $U = u$ and $V = 1/v$ and set $x = v$. After all these coordinate changes the singularity is located at $x = 0$ and the near singularity geometry is

$$ds^2 = -4w^{-\frac{3(6-p)}{10-2p}} v^{-2} du dv + w^{-\frac{6-p}{10-2p}} (d\rho^2 + ds^2(\mathbb{R}^p)) + w^{-\frac{p-2}{10-2p}} d\Omega_{7-p}^2, \quad (3.29)$$

where $w = \frac{10-2p}{2}(u + \frac{1}{v})$. Using the definition of the near singularity geometry, one can immediately read the exponents as

$$p = -\frac{2-p}{10-2p}, \quad w_1 = w_2 = \frac{6-p}{10-2p}, \quad q = \frac{2-p}{10-2p}, \quad p < 5 \quad (3.30)$$

For $p = 5$, the transformation to conformal coordinates is $r = e^w = e^{U+V}$. The singularity is located at either $U = -\infty$ or $V = -\infty$ and the two cases can be treated

symmetrically. If the singularity is located at $V = -\infty$, we define new conformal coordinates $v = e^V, u = e^U$. Using these, one can easily read the exponents as

$$p = \frac{1}{2}, \quad w_1 = w_2 = \frac{1}{2}, \quad q = \frac{3}{2}. \quad (3.31)$$

As another example of intersecting brane configuration in M-theory, one can take two orthogonally intersecting membranes at a 0-brane. The metric is

$$ds^2 = H_1^{\frac{1}{3}} H_2^{\frac{1}{3}} \left(-H_1^{-1} H_2^{-1} dt^2 + H_1^{-1} ds^2(\mathbb{R}^2) + H_2^{-1} ds^2(\mathbb{R}^2) + dr^2 + r^2 d\Omega_5^2 \right) \\ H_1 = 1 + \frac{Q_1}{r^4}, \quad H_2 = 1 + \frac{Q_2}{r^4}, \quad (3.32)$$

where H_1, H_2 are the harmonic functions associated with the membranes. Setting again $Q_1 = Q_2 = 1$, the near singularity geometry is

$$ds^2 = -r^{\frac{16}{3}} dt^2 + r^{\frac{4}{3}} (ds^2(\mathbb{R}^2) + ds^2(\mathbb{R}^2)) + r^{-\frac{8}{3}} dr^2 + r^{-\frac{2}{3}} d\Omega_5^2. \quad (3.33)$$

Changing coordinates as $r = w^{-\frac{1}{3}}$, the above metric can be rewritten as

$$ds^2 = w^{-\frac{16}{9}} (-dt^2 + \frac{1}{9} dw^2) + w^{-\frac{4}{9}} (ds^2(\mathbb{R}^2) + ds^2(\mathbb{R}^2)) + w^{\frac{2}{9}} d\Omega_5^2. \quad (3.34)$$

In the conformal coordinates $U = \frac{1}{2}(t + z)$, $V = \frac{1}{2}(-t + z)$, $z = \frac{1}{3}w$, the singularity at $r = 0$ is either located at $U = +\infty$ or $V = +\infty$. In the latter case, we perform another conformal transformation $U = u$ and $V = 1/v$ to locate the singularity at the origin. Setting $x = v$, it is easy to show that the exponents are

$$p = -\frac{2}{9}, \quad w_1 = w_2 = \frac{4}{9}, \quad q = -\frac{2}{9}. \quad (3.35)$$

Another example is the magnetic dual configuration of two M5-branes intersecting at the 3-brane. The metric is

$$ds^2 = H_1^{\frac{2}{3}} H_2^{\frac{2}{3}} \left(H_1^{-1} H_2^{-1} ds^2(\mathbb{R}^{1,3}) + H_1^{-1} ds^2(\mathbb{R}^2) + H_2^{-1} ds^2(\mathbb{R}^2) + dr^2 + r^2 d\Omega_2^2 \right) \\ H_1 = 1 + \frac{Q_1}{r}, \quad H_2 = 1 + \frac{Q_2}{r}, \quad (3.36)$$

Setting again $Q_1 = Q_2 = 1$, the near singularity geometry is

$$ds^2 = r^{\frac{2}{3}} ds^2(\mathbb{R}^{1,3}) + r^{-\frac{1}{3}} (ds^2(\mathbb{R}^2) + ds^2(\mathbb{R}^2)) + r^{-\frac{4}{3}} dr^2 + r^{\frac{2}{3}} d\Omega_2^2. \quad (3.37)$$

In this case the transformation to conformal coordinates is $r = z = e^{U+V}$. The singularity at $r = 0$ is either at $U = -\infty$ or $V = -\infty$. The two cases can be treated symmetrically by using the conformal transformation $v = e^V$ and $u = e^U$ to write the equation for the singularity as $uv = 0$. In the case where the singularity is located at $v = 0$, set $x = v$ and the exponents are

$$p = -\frac{1}{3}, \quad w_1 = w_2 = -\frac{1}{3}, \quad q = \frac{2}{3}. \quad (3.38)$$

The other case gives the same exponents.

4 POWER-LAW SINGULARITIES, STRINGS AND M-THEORY

In string theory and M-theory novel mechanisms have been proposed to resolved space-time singularities. Examples of such mechanisms are the resolution of orbifold singularities using the twisted sectors [39], the resolution of conifold singularities using D-branes [40] and the resolution of the singularities of planar Dp-brane solutions by lifting them to M-theory [41]. So far such mechanisms have not been extended to the context of black hole and cosmological singularities. Although there is no concrete proposal how to resolve such singularities, there are mechanisms within string theory that may allow a resolution. For example, the higher curvature corrections of string theory and M-theory can resolve singularities; for such a proposal see eg [42]. However in very few simple backgrounds such corrections have been computed because of the lack of understanding of α' corrections to all orders. Therefore, it is useful

- to be able to identify and interpret the nature of a singularity in a string background, ie whether the singularity is due to a brane placed in the background or to another object that has an interpretation within string theory and
- to have a criterion to estimate the severity of a spacetime singularity and the likelihood that such singularity can be resolved in string theory.

We propose that the near singularity geometries defined in the previous sections provide a way to identify the local nature of a singularity in a generic string background. One expects that as one goes near the singularity, the leading contribution to the metric and fluxes will be due to the matter placed at the singularity and the rest of the space will contribute to subleading terms⁴. Since the near singularity geometry is constructed by the leading contribution, it is expected that one can identify the local nature of the singularity by looking at the near singularity geometry. For example, if the power-law singularity of a supergravity background is the same as that of a planar D-brane, then one can conclude that this singularity is due to a planar D-brane. Clearly this argument can be extended to other branes and other objects in string theory that are singular in the effective theory and their solutions are known. In this way, we can have an understanding of the nature of singularities that occur in a particular background.

Having identified and interpreted a singularity in a generic string background, one can appeal to various mechanisms in string theory and M-theory to attempt to resolved it. For example, if a singularity is due to Dp-branes, one can conclude that near the singularity it is more appropriate to use gauge theory to describe the theory and if a

⁴The argument below applies only in this case.

singularity is due to a planar fundamental string, then one expects such a singularity to exist because of the presence of a fundamental object in the theory.

Using the arguments above and the fact that string theory is solvable in some Penrose limit plane waves, we can give an estimate about the severity of certain singularities and the likelihood that these can be resolved within string theory. One may expect that the Penrose limits of a background can be taken in a regular way which means that if string theory is well-defined in a supergravity background⁵, then it will be well-defined at all its Penrose limits. Since free strings behave well at mild and marginal ($\omega^2 < 1/4$) plane wave singularities, it is an indication that singularities for which all their Penrose limits are of mild or marginal ($\omega^2 < 1/4$) type may be resolvable within string theory. The above arguments are not a proof that string theory is consistent in backgrounds with Penrose limits that exhibit mild or marginal ($\omega^2 < 1/4$) singularities. It is only an indication that it may be so. Other consistency checks should also be considered like for example strong string coupling effects, backreaction and instantons. We have mostly focus on the singularities of the metric. The singularities of the other fields should also be analyzed as well.

Let us now investigate the case of backgrounds with timelike and null power-law singularities associated with plane waves, via a Penrose limit, which have marginal ($\omega^2 > 1/4$) or severe type singularities. There are such supergravity backgrounds, for example there are plane wave solutions with such profiles. One can argue that string theory cannot resolve such singularities. (We shall discuss the issue of higher curvature corrections at the end of the section.) As we have mentioned free string propagation is singular in plane waves with singularities of marginal ($\omega^2 > 1/4$) or severe type. Assuming that there are no objects in string theory that their timelike or null near singularity geometries have Penrose limits of the severe type, there does not seem to be an interpretation of such singularities in string theory and so there are no intrinsic string mechanisms that they can be used to resolve the singularity. It is known for example that for all branes the associated plane waves are of marginal type with $\omega^2 \leq 1/4$ [12, 13, 14] (see also appendix A). From these one can conclude that timelike and null power-law singularities which admit a Penrose limit for which the associated plane wave has a singularity of marginal ($\omega^2 > 1/4$) and severe type cannot be resolved in string theory. It is likely that a similar result holds for backgrounds with spacelike singularities.

It remains to investigate whether higher curvature corrections, like α' corrections, can resolve spacetime singularities. The plane wave backgrounds do not receive α' corrections [43, 44, 45, 46], so one does not expect a singularity to be resolved *after* taking the Penrose limit. As we have explained in appendix B, there are several ways to adapt the

⁵We shall discuss the effect of higher curvature terms at the end of the section.

Penrose limit after α' corrections are taken into account. It is clear though that it is not possible to deduce from the Penrose limit of a singular supergravity solution whether the associated string background, after all α' -corrections are taken into account, is singular or not. However it is curious that in the case that the corrected metric depends analytically on α' , the limit that preserves the homogeneity of the α' -corrected field equations (see appendix B) gives the same singular plane wave as that of the Penrose limit of the singular supergravity background. This suggests that either in the analytic case the singularity cannot be resolved with α' corrections⁶ or the singular plane waves are limits in the space of deformations of such string backgrounds which preserve the field equations. These suggest that consistency of string theory in such background would require consistency at the plane wave limit.

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⁶There does not seem to exist an example in the literature of a singular supergravity background for which all α' corrections are known and the corrected background is smooth.

A PENROSE LIMITS FOR POWER-LAW SINGULARITIES

There are three ways to compute the Penrose limits [47, 48, 49] of a spacetime. The first method was originally proposed by Penrose and uses adapted coordinates. This has recently been improved after the observation that one of the adapted coordinates is the Hamilton-Jacobi function for null geodesics [15]. The second method uses a covariant definition of the Penrose limit which identifies the plane wave profile with either a certain component of the Riemann tensor evaluated on a parallel transported frame along the null geodesic or with the frequencies of the geodesic deviation equation for null geodesics [10, 11]. The third method is a combination of the two methods above. It utilizes the Hessian of the Hamilton-Jacobi function evaluated on a parallel transported frame along the null geodesic. In particular, one defines the matrix

$$B_{ab} = E_a^\mu E_b^\nu \nabla_\mu \partial_\nu S, \quad g^{\mu\nu} \partial_\mu S \partial_\nu S = 0, \quad \mu, \nu = 0, \dots, D-1 \quad (\text{A.1})$$

where ∇ is the Levi-Civita connection, S is the Hamilton-Jacobi function and E_+, E_-, E_a , $a, b = 1, \dots, D-2$, is a parallel transported pseudo-orthonormal coframe along the null geodesic. Note that $\text{tr} B = \nabla^\mu \partial_\mu S$. The wave profile is given by

$$A_{ab} = \dot{B}_{ab} + \sum_c B_{ac} B_{cb}, \quad (\text{A.2})$$

where the derivative is with respect to the affine parameter, u , of the null geodesic. The associated plane wave at the limit is

$$ds^2 = 2dudv + A_{ab}(u)z^a z^b + (dz^a)^2. \quad (\text{A.3})$$

The description of all these methods is given in [11]. Here we shall use the first and third methods to compute the Penrose limits of the near singularity geometries described in section two.

A.1 SPACELIKE AND TIMELIKE SINGULARITIES

Motivated by the form of near singularity geometries of brane configurations in section three, we shall focus on the Penrose limits of the metrics

$$ds^2 = -2x^p dudv + \sum_i x^{w_i} ds^2(\mathbb{R}^{n_i}) + x^q d\Omega_m^2, \quad x = ku + \ell v, \quad (\text{A.4})$$

where $k, \ell = \pm 1, 0$ and the dimension of spacetime is $D = \sum_i n_i + m + 2$. We shall first consider the case where $k\ell \neq 0$. In this case, we define a new coordinate $y = ku - \ell v$ and rewrite the metric as

$$ds^2 = \frac{1}{2}\eta x^p (-dx^2 + dy^2) + \sum_i x^{w_i} ds^2(\mathbb{R}^{n_i}) + x^q d\Omega_m^2, \quad \eta = k\ell. \quad (\text{A.5})$$

The singularity is spacelike (timelike) for $\eta = 1$ ($\eta = -1$).

Using the rotation invariance of the above metric, we can write the Hamilton-Jacobi function S as

$$S = X(x) + Py + J_i z^i + L\theta , \quad (\text{A.6})$$

where $z^i = z^{i1}$ and $\{z^{ir}; r = 1, \dots, n_i\}$ are the standard coordinates in \mathbb{R}^{n_i} , and P , J_i and L are the conserved momenta associated with the translation invariance along y and z^i , and the rotational invariance along the angle θ of the sphere, respectively. The function X is determined by solving the Hamilton-Jacobi equation for null geodesics giving

$$\left(\frac{d}{dx}X(x)\right)^2 = P^2 + \frac{1}{2}\eta J_i^2 x^{-w_i+p} + \frac{1}{2}\eta L^2 x^{-q+p} . \quad (\text{A.7})$$

The non-trivial null geodesic equations are

$$\begin{aligned} (\dot{x})^2 &= 4x^{-2p} \left(\frac{d}{dx}X(x)\right)^2 = 4P^2 x^{-2p} + 2\eta J_i^2 x^{-w_i-p} + 2\eta L^2 x^{-q-p} , & \eta = \pm 1 \\ \dot{y} &= 2\eta x^{-p} P \\ \dot{z}^i &= x^{-w_i} J_i \\ \dot{\theta} &= x^{-q} L . \end{aligned} \quad (\text{A.8})$$

For $\eta \neq 0$, there are two distinct classes of null geodesics to consider. The first class are those null geodesics for which $J_i = 0$. In this case the geodesic equations are

$$\begin{aligned} (\dot{x})^2 &= 4x^{-2p} \left(\frac{d}{dx}X(x)\right)^2 = 4P^2 x^{-2p} + 2\eta L^2 x^{-q-p} , & \eta = \pm 1 \\ \dot{y} &= 2\eta x^{-p} P \\ \dot{z}^i &= 0 \\ \dot{\theta} &= x^{-q} L . \end{aligned} \quad (\text{A.9})$$

The analysis of the Penrose limits in this case resembles that already done in [11]. The other class of null geodesics is that for which one or more of the conserved charges $J_i \neq 0$.

A.1.1 The $J_i = 0$ case

The leading behaviour of the x geodesic equation in this case is

$$\begin{aligned} 1. \quad \dot{x} &= 2x^{-p} \tilde{P} , & p \geq q ; \quad \tilde{P} = P \ (p > q) ; \quad \tilde{P}^2 = P^2 + \frac{1}{2}\eta L^2 \quad (p = q) \\ 2. \quad \dot{x} &= \sqrt{2} x^{-\frac{p+q+2}{2}} L , & q > p , \quad \eta = -1 \end{aligned} \quad (\text{A.10})$$

where $\tilde{P}^2, \tilde{L}^2 \geq 0$ and we have chosen the plus sign in the equation for x . These equations can be solved as

$$\begin{aligned} 1. \quad & x^{p+1} = 2(p+1)\tilde{P} u, \quad p+1 > 0 \\ 2. \quad & x^{\frac{p+q+2}{2}} = \frac{p+q+2}{\sqrt{2}} L u, \quad p+q+2 > 0, \end{aligned} \quad (\text{A.11})$$

where the inequalities in the exponents arise from the requirement that the singularity at $x = 0$ is reached in finite affine time u .

Because of the symmetries that preserve both the metric and the choice of null geodesic, we expect that B and so A are diagonal. The wave profile can be easily be computed from

$$\begin{aligned} B_{ri,js} &= E_{ir}^\mu E_{js}^\nu \nabla_\mu \partial_\nu S = \delta_{ij} \delta_{rs} \partial_u \log g_{ir,js}^{\frac{1}{2}} = \delta_{ij} \delta_{rs} \partial_u \log x^{\frac{w_i}{2}} \\ B_{\hat{\alpha}\hat{\beta}} &= E_{\hat{\alpha}}^\mu E_{\hat{\beta}}^\nu \nabla_\mu \partial_\nu S = \delta_{\hat{\alpha}\hat{\beta}} \partial_u \log g_{\hat{\alpha}\hat{\alpha}}^{\frac{1}{2}} = \delta_{\hat{\alpha}\hat{\beta}} \partial_u \log(x^{\frac{q}{2}} \sin \theta), \quad \hat{\alpha}, \hat{\beta} = 2, 3, \dots, m \\ B_{11} &= \frac{1}{\sqrt{g}} \partial_\mu (g^{\mu\nu} \sqrt{g} \partial_\nu S) - \sum_i n_i B_{i1,i1} - (m-1) B_{22} = \partial_u \log(\dot{x} x^p x^{\frac{q}{2}}), \end{aligned} \quad (\text{A.12})$$

where $\dot{x} = \frac{d}{du} x$. Using the formulae for the metric and the expression of the wave profile in terms of B , we get

$$\begin{aligned} A_{ri,js} &= \delta_{ij} \delta_{rs} \frac{\partial_u^2 g_{ir,js}^{\frac{1}{2}}}{g_{ir,js}^{\frac{1}{2}}} = \delta_{ij} \delta_{rs} \frac{\partial_u^2 x^{\frac{w_i}{2}}}{x^{\frac{w_i}{2}}} \\ A_{\hat{\alpha}\hat{\beta}} &= \delta_{\hat{\alpha}\hat{\beta}} \frac{\partial_u^2 g_{\hat{\alpha}\hat{\alpha}}^{\frac{1}{2}}}{g_{\hat{\alpha}\hat{\alpha}}^{\frac{1}{2}}} = \delta_{\hat{\alpha}\hat{\beta}} \frac{\partial_u^2 (x^{\frac{q}{2}} \sin \theta)}{x^{\frac{q}{2}} \sin \theta} = \delta_{\hat{\alpha}\hat{\beta}} \left(\frac{\partial_u^2 x^{\frac{q}{2}}}{x^{\frac{q}{2}}} - \frac{L^2}{x^{2q}} \right) \\ A_{11} &= \frac{\partial_u^2 (\dot{x} x^p x^{\frac{q}{2}})}{\dot{x} x^p x^{\frac{q}{2}}}. \end{aligned} \quad (\text{A.13})$$

It is easy to see that a non-homogeneous plane wave occur when the Penrose limit is taken along null geodesics which exhibit behaviour 2, ie $x(u) \sim u^a$ for $a = \frac{2}{p+q+2}$ ($q > p$), and $q > p+2$.

We shall now investigate the behaviour of the null geodesics near the singularity and that of the associated frequencies. We begin with spacelike singularities, $\eta = 1$. Writing the solution for null geodesics as $x \sim u^a$, the various cases that we consider are described in the table below:

Conditions on (P, L)	Constraints on (p, q)	Behaviour	a
i. $P \neq 0, L = 0$	$p > -1$	1	$\frac{1}{p+1}$
ii. $P = 0, L \neq 0$	$p + q > -2$	2	$\frac{2}{p+q+2}$
iii. $P \neq 0, L \neq 0$	$p > q, p > -1$	1	$\frac{1}{p+1}$
iv. $P \neq 0, L \neq 0$	$p < q, p + q > -2$	2	$\frac{2}{p+q+2}$
v. $P \neq 0, L \neq 0$	$p = q > -1$	$1 = 2$	$\frac{1}{p+1}$

(A.14)

It turns out that the components $A_{ir,js} \sim u^{-2}$ and $A_{11} \sim u^{-2}$ but $A_{\hat{\alpha}\hat{\beta}} \sim u^{-\gamma}$, $\gamma \geq 2$. The frequency squares ω_i^2 and ω_α^2 , $\alpha = 1, \dots, m$, where $A_{ir,js} = -\delta_{ij}\delta_{rs}\omega_i^2 u^{-2}$, $A_{11} = -\omega_1^2 u^{-2}$ and $A_{\hat{\alpha}\hat{\beta}} = -\delta_{\hat{\alpha}\hat{\beta}}\omega_{\hat{\alpha}}^2 u^{-\gamma}$, are as follows:

(i) $P \neq 0, L = 0$. The frequencies are

$$\begin{aligned}\omega_i^2 &= \frac{w_i}{2(p+1)} \left(1 - \frac{w_i}{2(p+1)}\right), \\ \omega_\alpha^2 &= \frac{q}{2(p+1)} \left(1 - \frac{q}{2(p+1)}\right), \quad \gamma = 2, \quad \alpha = 1, \dots, m.\end{aligned}\quad (\text{A.15})$$

(ii) $P = 0, L \neq 0$. The frequencies are

$$\begin{aligned}\omega_i^2 &= \frac{w_i}{(p+q+2)} \left(1 - \frac{w_i}{(p+q+2)}\right), \\ \omega_1^2 &= \frac{p}{(p+q+2)} \left(1 - \frac{p}{(p+q+2)}\right), \\ \omega_\alpha^2 &= \begin{cases} \frac{q}{(p+q+2)} \left(1 - \frac{q}{(p+q+2)}\right), & q < p+2, \quad \gamma = 2 \\ \frac{1}{4} + \frac{1}{2q^2}, & q = p+2, \quad \gamma = 2 \\ 2^{\frac{2q}{p+q+2}} \frac{L^{\frac{2p-2q+4}{p+q+2}}}{(p+q+2)^{\frac{4q}{p+q+2}}}, & q > p+2, \quad \gamma = \frac{4q}{p+q+2}. \end{cases}\end{aligned}\quad (\text{A.16})$$

(iii) $P \neq 0, L \neq 0, (p > q)$. The frequencies are as in the case (i) but the case $p+1 = q$ with frequencies $\omega_1^2 = \omega_\alpha^2 = \frac{1}{4}$ does not arise because $p+1 > q$.

(iv) $P \neq 0, L \neq 0, p < q$. The frequencies ω_i^2 and ω_α^2 are as in the case (ii) but in addition $p < q$.

(v) $P \neq 0, L \neq 0, p = q$. The frequencies in this case are as in case (i) after setting $p = q$.

Next let us turn to the case of timelike singularities, $\eta = -1$. The various behaviours are summarized in the table below:

Conditions on (P, L)	Constraints on (p, q)	Behaviour	a
<i>i.</i> $P \neq 0, L = 0$	$p > -1$	1	$\frac{1}{p+1}$
<i>ii.</i> $P = 0, L \neq 0$		—	
<i>iii.</i> $P \neq 0, L \neq 0$	$p > q, p > -1$	1	$\frac{1}{p+1}$
<i>iv.</i> $P \neq 0, L \neq 0$	$p < q$	—	
<i>v.</i> $\sqrt{2} P > L $	$p = q > -1$	$1 = 2$	$\frac{1}{p+1}$

(A.17)

The frequencies are as follows:

(i) $P \neq 0, L = 0$. The frequencies are the same as those in case (i) for $\eta = +1$.

- (iii) $P \neq 0, L \neq 0, p > q$. The frequencies are the same as those in case (iii) for $\eta = +1$.
(v) $P \neq 0, L \neq 0, \sqrt{2}|P| > |L|, p = q$. The frequencies are the same as those in case (v) for $\eta = +1$.

The cases (ii) and (iv) do not occur because the null geodesics do not enter the singularity. We have verified that the substituting the exponents for the D6-brane, we recover the wave profile which has been computed in [12, 13, 14].

A.1.2 The $J_i \neq 0$ case

The leading behaviour of the null geodesics in terms of the affine parameter u near the singularity $x = 0$ depends on the various of the exponents and there are several cases to consider. In particular the leading behaviour of the x geodesic equation is

$$\begin{aligned} 1. \quad \dot{x} &= 2x^{-p} \tilde{P} & p \geq w_i, q, \quad \tilde{P}^2 > 0 \\ 2. \quad \dot{x} &= \sqrt{2} x^{-\frac{p+w_i}{2}} \tilde{J}_i & w_i \geq p, q, \quad \tilde{J}_i^2 > 0 \\ 3. \quad \dot{x} &= \sqrt{2} x^{-\frac{p+q}{2}} \tilde{L} & q \geq w_i, p, \quad \tilde{L}^2 > 0. \end{aligned} \quad (\text{A.18})$$

In case (1) (i) $\tilde{P} = P$, if $p > w_i, q$ ($\eta = \mp 1$), (ii) $\tilde{P} = \sqrt{P^2 + \frac{1}{2}\eta J_i^2}$, if $p = w_i > q$, (iii) $\tilde{P} = \sqrt{P^2 + \frac{1}{2}\eta L^2}$, if $p = q > w_i$ and (iv) $\tilde{P} = \sqrt{P^2 + \frac{1}{2}\eta J_i^2 + \frac{1}{2}\eta L^2}$, if $p = w_i = q$. In case (2), (i) $\tilde{J}_i = J_i$, if $w_i > p, q$ ($\eta = +1$), (ii) $\tilde{J}_i = \sqrt{J_i^2 + L^2}$, if $w_i = q > p$ ($\eta = +1$) and the rest of the possibilities are as in case (1). In case (3), (i) $\tilde{L} = L$, if $q > w_i, p$ ($\eta = +1$) and the rest of the possibilities are as either in case (1) or case (2). The rest of the geodesic equations remain the same. (We have chosen the plus sign in the equation for x).

The geodesic equations (A.18) can be solved to yield

$$\begin{aligned} 1. \quad x^{p+1} &= 2(p+1)\tilde{P}u & p > -1 \\ 2. \quad x^{\frac{p+w_i}{2}+1} &= \sqrt{2} \tilde{J}_i \left(\frac{p+w_i}{2} + 1 \right) u & p + w_i > -2 \\ 3. \quad x^{\frac{p+q}{2}+1} &= \sqrt{2} \tilde{L} \left(\frac{p+q}{2} + 1 \right) u & p + q > -2. \end{aligned} \quad (\text{A.19})$$

The inequality restrictions on the exponents arise from the requirement that the space-time is geodesically incomplete at $x = 0$, ie the singularity at $x = 0$ is reached in finite affine time u .

In the $J_i \neq 0$ case, it is not apparent that the wave profile A is diagonal as the choice of null geodesic breaks some of the translational symmetries in the \mathbb{R}^{n_i} directions. Because of this, it is rather involved to compute the wave profile with the method we have used for the $J_i = 0$ case. Instead, we shall proceed with adapted (Penrose) coordinates and

the Hamilton-Jacobi function to give the plane waves that arise in the Penrose limits in Rosen coordinates. The method has been explained in detail in [11]. For this first observe that the non-trivial geodesic equations can be solved formally as

$$\begin{aligned}
y &= Y(u, x_0) + y_0, & Y(u, x_0) &= 2\eta \int^u d\lambda x^{-p}(\lambda, x_0)P \\
R(x) &= u + R(x_0), & R(x) &= \frac{1}{2} \int dx x^p \left(\frac{d}{dx}X(x)\right)^{-1} \\
z^i &= Z^i(u, x_0) + z_0^i, & Z^i(u, x_0) &= J_i \int^u d\lambda x^{-w_i}(\lambda, x_0) \\
\theta &= \Theta(u, x_0) + \theta_0, & \Theta(u, x_0) &= L \int^u d\lambda x^{-q}(\lambda, x_0)
\end{aligned} \tag{A.20}$$

where y_0, x_0, z_0^i and θ_0 are integration constants. The adapted coordinates are $U = u$,

$$V = S(y_0, x_0, z_0^i, \theta_0) = Py_0 + X(x_0) + J_i z_0^i + L\theta_0 \tag{A.21}$$

and the rest of the integration constants subject to a gauge fixing condition. It is convenient to chose as the gauge fixing condition

$$Py_0 + J_i z_0^i + L\theta_0 = 0. \tag{A.22}$$

After taking the Penrose limit and doing some computation, we find that the plane wave in Rosen coordinates is

$$\begin{aligned}
ds^2 &= 2dUdV - \eta \frac{1}{2P^2} x^p(U) (J_i dz_0^i + Ld\theta_0)^2 + \sum_i x^{w_i}(U) ds^2(\mathbb{R}^{n_i}) \\
&+ x^q(U) d\theta_0^2 + x^q(U) \sin^2(\theta(U)) \sum_{\alpha>1}^m d\phi_0^\alpha,
\end{aligned} \tag{A.23}$$

where ϕ_0^α are the integration constants of the null geodesic equations associated with the rest of the angular coordinates that parameterize the m-sphere. We have assumed that $P \neq 0$. If $P = 0$ and $L \neq 0$, we can use the same gauge fixing condition but now we solve it for θ_0 . If both $P = L = 0$, we solve the gauge fixing condition with respect to one of the z_0^i for which the associated momentum $J_i \neq 0$. The plane wave metrics in the various cases above can be easily computed and we shall not present them here.

It is clear that the metric in Rosen coordinates (A.23) is off-diagonal and so the transformation to Brinkmann coordinates can be rather involved. This transformation is equivalent to the construction of the parallel frame along the null geodesic. Because of the non-diagonal nature of the metric, one expects that the solution for the parallel transported frame will in general involve path-ordered-exponentials. If the Rosen coordinates metric admits an additional isometry, then the associated plane wave is a homogeneous space. It may be that some of Penrose limits that arise are singular homogeneous plane waves with non-vanishing rotation as those of [22].

A.2 NULL SINGULARITIES

In the null case, we have $k\ell = 0$. If $k \neq 0$, the metric, after a possible change of coordinates, can be written as

$$ds^2 = 2x^p dudv + \sum_i x^{w_i} ds^2(\mathbb{R}^{n_i}) + x^q d\Omega_m^2, \quad x = ku. \quad (\text{A.24})$$

This can be rewritten in terms of the x coordinate as

$$ds^2 = 2x^p dx dv + \sum_i x^{w_i} ds^2(\mathbb{R}^{n_i}) + x^q d\Omega_m^2, \quad (\text{A.25})$$

after replacing v with kv . The case where $k = 0$ and $\ell \neq 0$ is symmetric and it will not be further explained. The Hamilton-Jacobi function is

$$S = -P^2 v + J_i z^i + L\theta + X(x), \quad (\text{A.26})$$

where

$$\frac{d}{dx}T = \left(\frac{J_i}{P}\right)^2 x^{p-w_i} + \left(\frac{L}{P}\right)^2 x^{p-q}. \quad (\text{A.27})$$

The equations for the null geodesics are

$$\begin{aligned} \dot{v} &= x^{-p} \frac{d}{du}T = \left(\frac{J_i}{P}\right)^2 x^{-w_i} + \left(\frac{L}{P}\right)^2 x^{-q} \\ \dot{z}^i &= x^{-w_i} J_i \\ \dot{\theta} &= x^{-q} L \\ \dot{x} &= -x^{-p} P^2. \end{aligned} \quad (\text{A.28})$$

The null geodesics reach the singularity $x = 0$ at finite affine time u provided that $p+1 > 0$.

The wave profile A is diagonal in this case. This can be seen by using adapted coordinates as in the case with $J_i \neq 0$ described in the previous section. In particular, there is always a gauge fixing condition such that the metric in Rosen coordinates is diagonal. It turns out that

$$\begin{aligned} B_{ir,js} &= \delta_{ij} \delta_{rs} \partial_u \log g_{i1,i1}^{\frac{1}{2}} \\ B_{\alpha\beta} &= \delta_{\alpha\beta} \partial_u \log g_{\alpha\alpha}^{\frac{1}{2}}, \quad \alpha, \beta = 1, \dots, m. \end{aligned} \quad (\text{A.29})$$

The wave profile is

$$\begin{aligned} A_{ir,js} &= \delta_{ij} \delta_{rs} \frac{\partial_u^2 g_{i1,i1}^{\frac{1}{2}}}{g_{i1,i1}^{\frac{1}{2}}} = \delta_{ij} \delta_{rs} \frac{\partial_u^2 x^{\frac{w_i}{2}}}{x^{\frac{w_i}{2}}} \\ A_{11} &= \frac{\partial_u^2 g_{11}^{\frac{1}{2}}}{g_{11}^{\frac{1}{2}}} = \frac{\partial_u^2 x^{\frac{q}{2}}}{x^{\frac{q}{2}}} \end{aligned}$$

$$A_{\hat{\alpha}\hat{\beta}} = \delta_{\hat{\alpha}\hat{\beta}} \frac{\partial_{\mathbf{u}}^2 g_{\hat{\alpha}\hat{\alpha}}^{\frac{1}{2}}}{g_{\hat{\alpha}\hat{\alpha}}^{\frac{1}{2}}} = \delta_{\hat{\alpha}\hat{\beta}} \left(\frac{\partial_{\mathbf{u}}^2 x^{\frac{\mathbf{q}}{2}}}{x^{\frac{\mathbf{q}}{2}}} - \frac{L^2}{x^{2\mathbf{q}}} \right) \quad \hat{\alpha}, \hat{\beta} = 2, \dots, m \quad (\text{A.30})$$

The components $A_{ir,js}$ and A_{11} of the wave profile behave as $A_{ir,js} \sim \mathbf{u}^{-2}$ and $A_{11} \sim \mathbf{u}^{-2}$ but $A_{\hat{\alpha}\hat{\beta}} \sim \mathbf{u}^{-\gamma}$, $\gamma \geq 2$. The frequency squares ω_i^2 and ω_{α}^2 , where $A_{ir,js} = -\omega_i^2 \delta_{ij} \delta_{rs} \mathbf{u}^{-2}$, $A_{11} = -\omega_1^2 \mathbf{u}^{-2}$ and $A_{\hat{\alpha}\hat{\beta}} = -\omega_{\hat{\alpha}}^2 \delta_{\hat{\alpha}\hat{\beta}} \mathbf{u}^{-\gamma}$, are as follows:

$$\begin{aligned} \omega_i^2 &= \frac{w_i}{2(\mathbf{p}+1)} \left(1 - \frac{w_i}{2(\mathbf{p}+1)} \right), \\ \omega_1^2 &= \frac{\mathbf{q}}{2(\mathbf{p}+1)} \left(1 - \frac{\mathbf{q}}{2(\mathbf{p}+1)} \right), \\ \omega_{\hat{\alpha}}^2 &= \begin{cases} \frac{\mathbf{q}}{2(\mathbf{p}+1)} \left(1 - \frac{\mathbf{q}}{2(\mathbf{p}+1)} \right), & \mathbf{q} \leq \mathbf{p}+1 \quad \gamma = 2 \\ \frac{1}{4} + \frac{L^2}{(\mathbf{p}+1)^2 P^4}, & \mathbf{q} = \mathbf{p}+1, \quad \gamma = 2 \\ \frac{L^2}{((\mathbf{p}+1)P^2)^{\frac{2\mathbf{q}}{\mathbf{p}+1}}}, & \mathbf{q} > \mathbf{p}+1, \quad \gamma = \frac{2\mathbf{q}}{\mathbf{p}+1}. \end{cases} \quad (\text{A.31}) \end{aligned}$$

In all the above case, we assume that $\mathbf{p} > -1$ for the null geodesics to reach the singularity $x = 0$ at finite affine time. Substituting the exponents of the Dp-branes, $\mathbf{p} \leq 5$, into (A.31), we have verified that they reproduce the of the Dp-branes frequency squares which have been computed using another method in [12, 13, 14]. This is also the case for the frequency squares of the fundamental string solution which has originally been computed in [12].

B α' -CORRECTIONS AND PENROSE LIMITS

In string theory and M-theory, the effective supergravity theories are modified by higher curvature terms. To distinguish between the field equations before or after higher curvature corrections are included we shall refer to the former as supergravity field equations and to the latter as string or M-theory field equations. String α' -corrections are accompanied with an appropriate power of α' and so the metric, dilaton and the various form-field strengths of generic solutions of the string are expected to depend on α' . For such backgrounds, there are three ways to take the Penrose limit:

1. One can take the Penrose limit of the original supergravity background before the α' corrections are included.
2. One can take the usual Penrose limit of the string background after the α' corrections are included.
3. One can take the usual Penrose limit which also involves a rescaling of the α' as $\alpha' \rightarrow \Omega^2 \alpha'$, where Ω is a parameter and $\Omega^2 \rightarrow 0$ at the limit [12].

The limit in case (1) is consistent in the sense that the associated plane wave will solve the string field equations. This is because plane wave backgrounds do not receive α' corrections [44, 45, 46]. The wave profile is the matrix of frequency squares of null geodesic deviation equation of the supergravity metric.

In case (2), the resulting plane wave will not necessarily solve the string field equations. This is because the string field equations are not homogeneous with respect to the Ω -scaling necessary to take the Penrose limit. The wave profile is the matrix of frequency squares of null geodesic deviation equation of the string metric and in general will be different from that of the associated supergravity metric.

In case (3), the limit is a modification of the Penrose limit. To take this modified limit, one first puts the string metric in adapted (Penrose) coordinates and then performs the usual rescaling of the coordinates with the parameter Ω , as for the Penrose limit. In addition one rescales α' as described. If the metric in adapted (Penrose) coordinates depends analytically of α' , the metric at the limit is the one computed in (1) from the associated supergravity background. The Penrose limit commutes with the operation of α' corrections. However, if the string metric does not depend analytically on α' , then either the limit is ill defined or it gives a metric which is not always a plane wave. If the limit is well-defined, the background at the limit will solve the string field equations because they transform homogeneously under the scaling required for (3).

To illustrate the features explained above consider the singular metric associated with the $SL(2, \mathbb{R})_k/U(1)$ coset model [50, 51, 52]

$$\begin{aligned} ds^2 &= \frac{1}{2}(k-2)(-dt^2 + dr^2 + \beta^2(r)d\theta^2) \\ \frac{4}{\beta^2} &= \tanh^2\left(\frac{r}{2}\right) - \frac{2}{k} \end{aligned} \quad (\text{B.1})$$

where the level $k \sim \alpha^{-1}$ and we have added a spectator time direction for the metric to have Lorentzian signature. This background has also non-vanishing dilaton but for simplicity we shall not consider this here. This metric is singular at $\tanh^2(\frac{r_0}{2}) = \frac{2}{k}$. The associated supergravity metric is found at the limit $k \rightarrow \infty$ as

$$ds^2 = \frac{k}{2}(-dt^2 + dr^2 + 4 \coth(\frac{r}{2})d\theta^2) . \quad (\text{B.2})$$

The singularity of the supergravity metric is at $r = 0$. We shall examine the Penrose limits of the near singularity geometries for the two metrics above.

To take the Penrose limit as described in case (1), it suffices to take the usual Penrose limit for the supergravity metric (B.2). It is easy to see that the near singularity geometry in this case is

$$ds^2 = \frac{k}{2}(-dt^2 + dx^2 + x^{-2}d\theta^2) . \quad (\text{B.3})$$

The plane wave profile has one independent component A which can be easily evaluated from the Laplacian of the Hamilton-Jacobi function of this background. A short computation reveals that

$$A = 2u^{-2} , \tag{B.4}$$

ie the frequency is $\omega^2 = -2$.

To take the limit as described in case (2), it suffices to take the usual Penrose limit for the metric (B.1). It is easy to see that the near singularity geometry in this case is

$$ds^2 = \frac{1}{2}(k-2)(-dt^2 + dx^2 + x^{-1}d\theta^2) . \tag{B.5}$$

A short computation reveals that

$$A = \frac{3}{4}u^{-2} , \tag{B.6}$$

ie the frequency is $\omega^2 = -\frac{3}{4}$. Therefore in both cases, the resulting plane waves are homogeneous but the frequencies are different.

Finally to take the limit as described in case (3), it suffices to observe that the dependence of the metric (B.1) in Penrose coordinates is analytic in α' . Therefore this limit gives the same plane wave as that computed in (i).

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